A New Look at the Perfectly Matched Layer (PML) Concept for the Reflectionless Absorption of Electromagnetic Waves

R. Mittra, *Fellow, IEEE*, and Ü. Pekel

***Abstract*—The Perfectly Matched Layer (PML) concept has been introduced recently by Berenger with the objective of developing an absorbing boundary condition for the Finite Difference Time Domain (FDTD) method. In its original formulation, the PML approach is based on the splitting of the field components into two sub-components which are weighted with different constituent parameters. The objectives of this paper are to take a critical look at the PML concept with a view to providing some interpretations of the same; to examine the Maxwellian nature of the PML layer; to present a version of the PML approach that does not involve the "splitting" and the resulting six components for each of the electric and magnetic fields in the PML layer; and, finally, to compare it with other approaches, e.g., co-ordinate scaling.**

I. Introduction

BERENGER [1] has recently introduced a novel concept for designing a "Perfectly Matched Layer" (PML), that provides reflectionless absorption of electromagnetic waves when they impinge upon this layer (also, see Katz *et al.* [2]). A close examination of the PML concept has raised several important questions that must be answered satisfactorily in order that a thorough understanding of this concept can be developed and refined versions of the PML layer can be synthesized. Some of these questions are:

1) Why is it necessary in the PML approach to decompose (split) each of the components of the electric and magnetic fields into two constituent parts at the cost of increasing the memory requirements?
2) Since two different constitutive parameters are assigned to each of these split field components, can the PML layer be represented in terms of the conventional complex $[\varepsilon]$ and $[\mu]$ tensors?
3) Are Maxwell's equations satisfied in the PML medium? If not, what equations do the PML fields satisfy?
4) Is the PML layer active or passive, i.e., does the PML layer have negative conductivities or implicit sources embedded in it?
5) Is the PML concept equivalent to co-ordinate scaling approach presented in [3], [4]?

Obviously, the answers to above questions are important so that we can be sure that the continuity equations, as dictated by Maxwell's curl equations, are still applicable at the interfaces between free-space and the PML layer, and that the field solution generated by using the PML is going to be stable as well as uncorrupted by spurious solutions. We hope to shed some light into the above questions through the exposition of the PML concept as presented below.

Manuscript received September 23, 1994.

The authors are with the Electromagnetic Communication Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA.

IEEE Log Number 9408308.

II. Reformulation of the PML Equations

The principal strategy of the PML approach is to devise a layer such that a plane wave incident at an arbitrary angle from the free-space upon a semi-infinite PML region is totally transmitted into the PML region without any reflection in the free space region. The original, split-form of the PML equations in the time-domain are given by [1]:

$$\mu_0 \frac{\partial H_{xy}}{\partial t} + \sigma_y^* H_{xy} = -\frac{\partial (E_{zx} + E_{zy})}{\partial y},$$
$$\mu_0 \frac{\partial H_{xz}}{\partial t} + \sigma_z^* H_{xz} = \frac{\partial (E_{yx} + E_{zy})}{\partial z} \tag{1a}$$
$$\mu_0 \frac{\partial H_{yz}}{\partial t} + \sigma_z^* H_{yz} = -\frac{\partial (E_{xy} + E_{xz})}{\partial z},$$
$$\mu_0 \frac{\partial H_{yx}}{\partial t} + \sigma_x^* H_{yx} = \frac{\partial (E_{zx} + E_{zy})}{\partial x} \tag{1b}$$
$$\mu_0 \frac{\partial H_{zx}}{\partial t} + \sigma_x^* H_{zx} = -\frac{\partial (E_{yx} + E_{yz})}{\partial x},$$
$$\mu_0 \frac{\partial H_{zy}}{\partial t} + \sigma_y^* H_{zy} = \frac{\partial (E_{xy} + E_{xz})}{\partial y} \tag{1c}$$
$$\varepsilon_0 \frac{\partial E_{xy}}{\partial t} + \sigma_y E_{xy} = \frac{\partial (H_{zx} + H_{zy})}{\partial y},$$
$$\varepsilon_0 \frac{\partial E_{xz}}{\partial t} + \sigma_z E_{xz} = -\frac{\partial (H_{yx} + H_{yz})}{\partial z} \tag{2a}$$
$$\varepsilon_0 \frac{\partial E_{yz}}{\partial t} + \sigma_z E_{yz} = \frac{\partial (H_{xy} + H_{xz})}{\partial z},$$
$$\varepsilon_0 \frac{\partial E_{yx}}{\partial t} + \sigma_x E_{yx} = -\frac{\partial (H_{zx} + H_{zy})}{\partial x}, \tag{2b}$$
$$\varepsilon_0 \frac{\partial E_{zx}}{\partial t} + \sigma_x E_{zx} = \frac{\partial (H_{yx} + H_{yz})}{\partial x},$$
$$\varepsilon_0 \frac{\partial E_{zy}}{\partial t} + \sigma_y E_{zy} = -\frac{\partial (H_{xy} + H_{xz})}{\partial y}. \tag{2c}$$

We note in the above equations that after the field components are split, their respective sub-components are "weighted" by different constitutive parameters, raising the question whether the PML medium can be described by conventional 3 × 3

1051-8207/95$04.00

constitutive parameter tensors. To answer this question we eliminate all of the split fields $(H_{xy}, E_{xy}, \cdots)$ from (1), (2) and replace them by the conventional ones $(H_x, E_x, \cdots)$ instead (for example, we solve (1a) to get $H_{xz} = -jk_zE_y/(\sigma_z^* + j\omega\mu_0))$. The unsplit equations read:

$$(j\omega\mu_0 + \sigma_1^*)H_x = -jk_zE_y + jk_yE_z + \frac{\sigma_2^* - \sigma_1^*}{j\omega\mu_0 + \sigma_2^*}jk_zE_y \tag{3a}$$

$$(j\omega\mu_0 + \sigma_2^*)H_y = -jk_xE_z + jk_zE_x + \frac{\sigma_3^* - \sigma_2^*}{j\omega\mu_0 + \sigma_3^*}jk_xE_z \tag{3b}$$

$$(j\omega\mu_0 + \sigma_3^*)H_z = -jk_yE_x + jk_xE_y + \frac{\sigma_1^* - \sigma_3^*}{j\omega\mu_0 + \sigma_1^*}jk_yE_x \tag{3c}$$

$$(j\omega\varepsilon_0 + \sigma_1)E_x = -jk_yH_z + jk_zH_y - \frac{\sigma_2 - \sigma_1}{j\omega\varepsilon_0 + \sigma_2}jk_zH_y \tag{4a}$$

$$(j\omega\varepsilon_0 + \sigma_2)E_y = -jk_zH_x + jk_xH_z - \frac{\sigma_3 - \sigma_2}{j\omega\varepsilon_0 + \sigma_3}jk_xH_z \tag{4b}$$

$$(j\omega\varepsilon_0 + \sigma_3)E_z = -jk_xH_y + jk_yH_x - \frac{\sigma_1 - \sigma_3}{j\omega\varepsilon_0 + \sigma_1}jk_yH_x \tag{4c}$$

where, for convenience, we have used a frequency domain representation, and the subscripts 1, 2, 3 of the electric and magnetic conductivities (σ's) can be associated with the y-, z-, and x-directions, respectively (i.e., $\sigma_1 \leftrightarrow \sigma_y, \sigma_2 \leftrightarrow \sigma_z, \sigma_3 \leftrightarrow \sigma_x$, etc $\cdots$). We note, immediately, that in addition to the regular terms arising from the Maxwell's curl equations in an anisotropic medium with a uniaxial, 3×3 conductivity tensor, there are terms involving the differences of the conductivities, e.g. $(\sigma_1 - \sigma_2)$, etc., appearing on the right-hand-sides of the above equations, which must be interpreted as source terms, albeit dependent on the excitation (i.e., the $\overline{E}$ and $\overline{H}$ fields), that are distributed in the entire semi-infinite PML region. We must therefore conclude that due to the presence of the distributed dependent sources on the right-hand-sides of (3a) to (4c), the PML medium ceases to be passive, even if the σ's are non-negative.

Next, we observe that if we impose the "PML impedance matching condition"

$$\frac{\sigma_1}{\varepsilon_0} = \frac{\sigma_1^*}{\mu_0}, \quad \frac{\sigma_2}{\varepsilon_0} = \frac{\sigma_2^*}{\mu_0}, \quad \frac{\sigma_3}{\varepsilon_0} = \frac{\sigma_3^*}{\mu_0} \tag{5}$$

then (3) and (4) become "dual" equations. Without loss of generality, we can represent the wave propagating in the PML medium as a linear combination of TE-to-z and TM-to-z modes, as defined in [5] by:

$$\begin{aligned}\overline{E}^{TE} &= (\hat{x}a_x^e \sin\phi - \hat{y}a_y^e \cos\phi)e^{-j(k_xx+k_yy+k_zz)}\\ \overline{H}^{TE} &= (-\hat{x}b_x^e \cos\theta\cos\phi - \hat{y}b_y^e \cos\theta\\ &\quad \cdot \sin\phi + \hat{z}b_z^e \sin\theta)e^{-j(k_xx+k_yy+k_zz)}\end{aligned} \tag{6a}$$

$$\begin{aligned}\overline{E}^{TM} &= (-\hat{x}a_x^m \cos\theta\cos\phi - \hat{y}a_y^m \cos\theta\\ &\quad \cdot \sin\phi + \hat{z}a_z^m \sin\theta)e^{-j(k_xx+k_yy+k_zz)}\\ \overline{H}^{TM} &= (-\hat{x}b_x^m \sin\phi + \hat{y}b_y^m \cos\phi)e^{-j(k_xx+k_yy+k_zz)}\end{aligned} \tag{6b}$$

where the superscripts "e" and "m" denote the TE and TM modes, respectively. In order to be able to match the fields at the interface of the free-space and the PML region, we must impose the following constraints on the complex coefficients $a_x^{e,m}, a_y^{e,m}, b_x^{e,m}$, and $b_y^{e,m}$ associated with the tangential x- and y-directions in (6):

$$a_x^{e,m} = a_y^{e,m}, \quad b_x^{e,m} = b_y^{e,m}, \quad a_x^{e,m} = -\eta_0 b_y^{e,m}. \tag{7}$$

Furthermore, as a consequence of the phase-matching condition which has to be satisfied at the interface between the two regions, the tangential wave numbers k_x and k_y in the PML region must be equal to their counterparts in the free-space region, which implies that $k_x = k_0 \sin\theta\cos\phi = k_{x0}, k_y = k_0 \sin\theta \sin\phi = k_{y0}$. In addition, to ensure a proper decay of the wave in the PML region along the preferred z-direction, and thereby produce the desired absorption effect, the wave number k_z in the PML region must have the form $k_z = k_0(1-jk_{ri})\cos\theta$. The positive-real number k_{ri} can be chosen such that a pre-specified value of attenuation is achieved for a given value of the transmission (= incidence) angle θ.

We now substitute the field representation in the PML region, as postulated in (6) in the form of a linear combination of TE-to-z and TM-to-z fields, into (3a) – (4c). After some algebraic manipulations, we arrive at the conditions:

$$1 - jk_{ri} = 1 - j\frac{\sigma_2}{\omega\varepsilon_0} = 1 - j\frac{\sigma_2^*}{\omega\mu_0} \tag{8}$$

$$\sigma_1 = \sigma_3 = 0, \qquad \sigma_1^* = \sigma_3^* = 0. \tag{9}$$

It may be verified that the above constraints are necessary in order for the field solution to simultaneously satisfy (6), (3), and (4). Of course, the σ values must be permuted as the direction of the normal is changed from z to x to y. Also, at the edges of the problem domain, where two faces come together, one may simply "overlap" the two constitutive parameters pertaining to the two preferred directions of propagation, while maintaining a zero conductivity in the third dimension. Likewise, at the corner regions of the problem domain, one may include all three conductivities in the governing equations since one is effectively faced with three directions of propagation. We should point out that the corner regions play a pivotal role, since the PML layer itself does *not* absorb any of the transverse component of the power entering into the layer, and this task must be handled by the corner regions.

Using (8), (9), the frequency-domain relations (3a) – (4c) for the PML medium may be rewritten in a simpler form as:

$$j\omega\mu_0H_x = jk_yE_z - jk_zE_y\frac{1}{1 - j\frac{\sigma_2^*}{\omega\mu_0}} \tag{10a}$$

$$j\omega\mu_0H_y = jk_zE_x\frac{1}{1 - j\frac{\sigma_2^*}{\omega\mu_0}} - jk_xE_z \tag{10b}$$

$$j\omega\mu_0H_z = jk_xE_y - jk_yE_x \tag{10c}$$

$$j\omega\varepsilon_0E_x = jk_zH_y\frac{1}{1 - j\frac{\sigma_2}{\omega\varepsilon_0}} - jk_yH_z \tag{11a}$$

$$j\omega\varepsilon_0 E_y = jk_x H_z - jk_z H_x \frac{1}{1 - j\frac{\sigma_2}{\omega\varepsilon_0}} \quad (11b)$$

$$j\omega\varepsilon_0 E_z = jk_y H_x - jk_x H_y. \quad (11c)$$

The above six equations are identical to those presented in a recent investigation of perfectly matched absorbing boundary conditions based on anisotropic lossy mapping of space [3], [4]. It is evident that (10), (11) can be obtained from conventional Maxwell's curl equations by introducing the complex spatial variable z', which is related to the spatial variable z by

$$z' = z\left(1 - j\frac{\sigma_2}{\omega\varepsilon_0}\right) = z\left(1 - j\frac{\sigma_2^*}{\omega\mu_0}\right) \quad (12)$$

assuming that the z-direction is the direction of propagation, and the x- and y-directions are the tangential directions in the PML medium. This leads us to conclude that the space-mapping procedure leads to non-Maxwellian fields as well (i.e., the fields do not satisfy the two standard Maxwell curl equations). In addition, we can readily show by using (9) in (10), (11) that the latter reduce to Maxwell's curl equations in *free-space*, which is a surprising result since we are dealing with a medium with a uniaxial conductivity, and the fields in this medium should *not* be satisfying Maxwell's equations in free-space.

Although we will not provide the details here because of lack of space, it is also possible to start the entire procedure outlined above by deleting the dependent source terms in (3), (4) at the outset, such that the above equations are entirely Maxwellian. If we require the field solution in the PML region to still have the same form as before, we obtain a totally different uniaxial medium, whose constitutive parameters are 3×3 (diagonal) matrices with *negative* σ_3 and σ_3^*, with $\sigma_1 = \sigma_2 > 0, \sigma_1^* = \sigma_2^* > 0$, and whose ε_3 and μ_3 are also different from ε_0 and μ_0. This result is consistent with that reported by Sacks *et al.* [6], and leads to the conclusion that a Maxwellian PML medium must be active with negative σ in the normal direction.

We are currently studying the problem of adapting the PML approach to FEM in the frequency domain. We find that this is not a particularly straightforward task if we desire to retain the conventional FEM formulation, using either the E or H fields only. This is because the usual curl-curl equations for either E or H can no longer be derived from (3), (4), and the presence of the source terms in these equations exacerbate the problem.[1] We should also be cautious when dealing with non-Maxwellian equations, or with media that have negative conductivities, since, to the best of our knowledge, the stability criteria for such media have not been established as yet, and we should be vigilant for the growth of round-off errors that can create long-term instability problems. We have, in fact, observed such instabilities for inhomogeneous, layered dielectric regions, and are currently investigating the same.

Before closing this discussion we should mention that while the forms of the equations in (3), (4), which are the unsplit versions of (1), (2), deal with fewer field components than the split version equations, they require second- rather than first-order time derivatives. This, of course, is also true of the scaled version of the PML equations given in (10), (11), which, as stated above, are identical to (3), (4).

[1] An approach to alleviating this problem has recently been reported by the authors [7], [8]

III. Conclusion

We draw the following conclusions on the basis of our investigation of the PML approach: 1) The PML equations of Berenger can be transformed into unsplit versions; 2) The PML equations are non-Maxwellian as they include dependent sources, that render the medium active; 3) The fields in the PML region have been shown to satisfy the Maxwell's equations corresponding to *free-space,* even though the medium is lossy and uniaxially conducting; 4) the same equations have been found to be identical to those obtained previously in a study based on the concept of anisotropic lossy mapping of space by coordinate stretching; 5) A Maxwellian PML-like medium, originally proposed by Sacks *et al.*, is different from the Berenger PML medium, not only because its conductivity tensor has the non-zero term in the normal direction, but also because this term is negative, suggesting an active medium of a different type; 6) the corner regions play a pivotal role in absorbing transversely-directed power in the PML layer and must be designed carefully; and, finally, 7) instabilities have been found when the region under investigation has layered inhomogeneities.

References

[1] J. P Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Comp. Phys.*, vol 114, no 2, pp 185–200, Oct 1994

[2] D. S. Katz, E. T. Thiele, and A. Taflove, "Validation and extension to three dimensions of the Berenger PML absorbing boundary condition for FDTD meshes," *IEEE Microwave and Guided Wave Lett.*, vol. 4, no. 3, pp. 268–270, Aug. 1994.

[3] C M. Rappaport, "Perfectly matched absorbing boundary conditions based on anisotropic lossy mapping of space," to be published.

[4] W C Chew and W H. Weedon, "A 3D perfectly matched medium from modified Maxwell's equations with stretched coordinates," *Microwave Opt. Tech. Lett.*, vol. 7, no 13, pp. 599–604, Sept 1994

[5] M. Born and E. Wolf, *Principles of Optics*. New York: McMillan, 1964.

[6] Z. S Sacks, D M. Kingsland, R. Lee, and J.-F. Lee, "A perfectly matched anisotropic absorber for use as an absorbing boundary condition," private communication.

[7] Ü Pekel and R Mittra, "A finite element method frequency domain application of the perfectly matched layer (PML) concept," to appear.

[8] ——, "Mesh truncation in the finite element frequency domain method with a perfectly matched layer (PML) applied in conjunction with analytic and numerical absorbing boundary conditions," to be published